Find the range of $f(x) = \log_2\left(\dfrac{x^2+4}{x^2+1}\right)$

Let $\quad u = \dfrac{x^2+4}{x^2+1} \Rightarrow ux^2 + u = x^2 + 4 \quad \Rightarrow (u-1)x^2 = 4 - u \Rightarrow x^2 = \dfrac{4-u}{u-1} \geq 0 \Rightarrow 1 < u \leq 4$

$\therefore \quad \log_2 1 < \log_2 u \leq \log_2 4 \Rightarrow 0 < y \leq 2$

$$u = \frac{x^2+4}{x^2+1} = 1 + \frac{3}{x^2+1}$$

$u_{\min} \approx \lim\limits_{x\to\infty} 1 + \dfrac{3}{x^2+1} = 1 + \dfrac{3}{\infty} = 1 \quad$ and $\quad u_{\max} = 1 + \dfrac{3}{(x^2+1)_{\min}} = 1 + \dfrac{3}{0+1} = 4$

$\therefore \quad 1 < u \leq 4 \Rightarrow 0 < y \leq 2$

Find the range of $y = \sqrt{\sin x} + \sqrt{\cos x} \;\; 0 \leq x \leq \dfrac{\pi}{3}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\sin x}}.\cos x + \frac{1}{2\sqrt{\cos x}}.(-\sin x)$$

$$= \frac{(\cos x)^{3/2} - (\sin x)^{3/2}}{2\sqrt{\sin x}.\sqrt{\cos x}} = 0 \;\Rightarrow\; x = \frac{\pi}{4}$$

Sign scheme for $\dfrac{dy}{dx}$

$0 \quad + \quad \pi/4 \quad - \quad \pi/3$

$f(0) = 1$

$f\left(\dfrac{\pi}{4}\right) = \sqrt{\dfrac{1}{\sqrt{2}}} + \sqrt{\dfrac{1}{\sqrt{2}}} = 2^{3/4}$

$f\left(\dfrac{\pi}{3}\right) = \sqrt{\dfrac{\sqrt{3}}{2}} + \sqrt{\dfrac{1}{2}} = \dfrac{3^{1/4}+1}{\sqrt{2}} > 1$

$\therefore \quad$ Range is $1 \leq y \leq 2^{3/4}$

A function $y = f(x)$ on domain 'D' is said to be periodic if there exists 'a' positive number 'T' such that $f(x + T) = f(x)\ \forall x \in D$. The least positive value of 'T' is called the fundamental period of the function $f(x)$.

In order to test the periodicity of $f(x)$ put $f(x + T) = f(x)$, find all possible values of 'T' independent of x. If no positive value of 'T' independent of 'x' is possible then $f(x)$ is said to be non-periodic or aperiodic. If positive value of 'T' independent of x is possible then $f(x)$ is said to be periodic and the period of $f(x)$ will be the least positive value of 'T'.

Pb.5 Which of the following functions are periodic ? What are their periods?

(i) $\cos x$ (ii) $|\cos x|$ (iii) $\cos\sqrt{x}$

(iv) $x - [x]$

Sol. (i) Let $f(x) = \cos x$

we put $f(x + T) = f(x)$

i.e. $\cos(x + T) = \cos x$

$\Rightarrow$ $x + T = 2n\pi \pm x$, $n = 0, \pm 1, \pm 2$ ....

$$\Rightarrow \quad T = \begin{cases} 2n\pi - 2x \ (\textit{not independent of } x) \\ \qquad\qquad or \\ 2n\pi \ \ (\textit{This is independent of } x) \end{cases}$$

$\therefore$ The positive value of T independent of 'x' is given by

$T = 2n\pi$, $n = 1, 2, 3, ...$

$\therefore$ The least positive value of $T = 2\pi$ (putting $n = 1$).

(ii) $f(x) = |\cos\ x|$

put $f(x + T) = f(x)$

i.e. $|\cos(x + T)| = |\cos x|$

$\Rightarrow$ $|\cos(x + T)|^2 = |\cos x|^2$

$\Rightarrow$ $\cos^2(x + T) = \cos^2 x\ [\because\ |x|^2 = x^2]$

$\Rightarrow$ $2\cos^2(x + T) = 2\cos^2 x$

$\Rightarrow$ $1 + \cos 2(x + T) = 1 + \cos 2x$

$\Rightarrow$ $\cos 2(x + T) = \cos 2x$

$\therefore$ $2(x + T) = 2n\pi \pm 2x$

$$\text{i.e.} \quad T = \begin{cases} n\pi - 2x (\text{not independent of } x) \\ \qquad\qquad \text{or} \\ \qquad\qquad n\pi \end{cases}$$

The positive value of 'T' independent of x are given by

$T = n\pi$, $n = 1, 2, 3,$ ....

$\therefore$ Least positive value of $T = \pi$

(iii) $f(x) = \cos\sqrt{x}$

Let $f(x + T) = f(x)$

$\Rightarrow$ $\cos\sqrt{x+T} = \cos\sqrt{x}$

$\therefore$ $\sqrt{x+T} = 2n\pi \pm \sqrt{x}$

or $\quad T = \left(2n\pi \pm \sqrt{x}\right)^2 - x$

Which is not independent of x. Hence, $f(x)$ is non periodic.

(iv) $\quad f(x) = x - [x]$

Let $f(x + T) = f(x)$

$\therefore \quad x + T - [x + T] = x - [x]$

$\Rightarrow \quad T = [x + T] - [x]$

$= \quad$ Integer – Integer = integer

$\therefore$ T is an integer and we know that least positive integer is 1.

$\therefore$ T = 1 Hence $f(x)$ is periodic with period '1'.

Student can also prove that $f(x) = \sqrt{x-[x]}$

$g(x) = \left(x-[x]\right)^2$ are also of the same period.

Characteristics of $f(x) = x-[x], \sqrt{x-[x]}$ or $\left(x-[x]\right)^2$

(i) These are periodic with period '1'.

(ii) Their range is $[0,1[$ i.e. $0 \le f(x) < 1$.

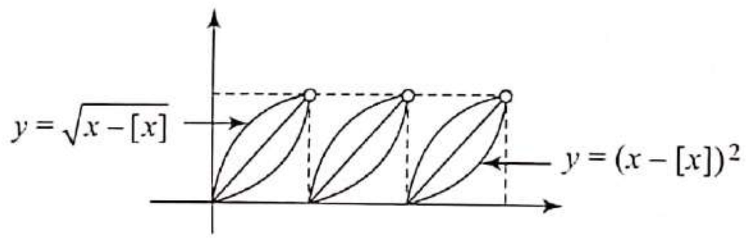
(iii) These are discontinuous at all integers.

For being more familiar see the graphs

$f(x) = x - [x]$

$$= \begin{cases} x-(-1) = x+1 & ;-1 \le x < 0 \\ x-0 = x & ;0 \le x < 1 \quad \text{and so on.} \\ x-1 & ;1 \le x < 2 \end{cases}$$

Graph of $f(x) = x - [x]$



2.8.1 Important Results

(i) $\sin^n x$, $\cos^n x$, $\sec^n x$, and $\text{cosec}^n x$ are periodic functions with period π when n is even and 2π when n is odd or fraction. *e.g.* period of $\sin^2 x$ is π but period of $\sin^3 x, \sqrt{\sin x}$ is 2π. But period is π if n = even/odd. e.g. $2/3$.

(ii) $\tan^n x$ and $\cot^n x$ are periodic functions with period π irrespective of 'n' .

(iii) $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$, $|\sec x|$, & $|\text{cosec}\, x|$ are periodic functions with period π.

(iv) If $f(x)$ is periodic with period T, then:

(a) $k.f(x)$ is periodic with period T.

(b) $f(x + b)$ is periodic with period T.

(c) $f(x) + c$ is periodic with period T.

(d) $f(ax)$ is periodic with period $\dfrac{T}{|a|}$.

(e) $kf(ax+b)$ is periodic with period $\frac{T}{|a|}$. Thus note that the period is affected only by coefficient of x, such as period of $\sin x$ is 2π but the period of $\left\{3\sin\left(2x+\frac{\pi}{9}\right)\right\}+5$ is equal to $\frac{2\pi}{2}=\pi$.

(v) Let $h(x) = af(x) \pm bg(x)$. If $f(x)$ and $g(x)$ are periodic functions with period T_1 & T_2 respectively then h(x) is also periodic and the period of $h(x)$ is the L.C.M. of T_1 & T_2.

Note: The result (v) is not always applicable

Pb.6 Find the period of

(a) $f(x)=\sin(2\pi x+\pi/4)+2\sin(3\pi x+\pi/3)$

(b) $f(x) = \cos^2 x + \sin^2 x$

Sol. (a) Period of $\sin(2\pi x+\pi/4) = 2\pi/2\pi = 1$

Period of $2\sin(3\pi x+\pi/3) = 2\pi/3\pi = 2/3$

Period of $f(x)$ = L.C.M. of 1 and 2/3

$$\frac{\text{L.C.M. of Numerator}}{\text{H.C.F. of Denominator}} = \frac{\text{L.C.M. of } 1 \,\&\, 2}{\text{H.C.F. of } 1 \,\&\, 3} = \frac{2}{1} = 2$$

(b) $\therefore$ Period of $\cos^2 x = \pi$ and

Period of $\sin^2 x = \pi$.

$\therefore$ Period of $f(x)$ should be L.C.M. of π & π.

i.e. π which is false.

Because $f(x) = \cos^2 x + \sin^2 x = 1$ which is a constant and costant functions are periodic functions having no fundamental period.

Another example,

$f(x) = |\cos x| + |\sin x|$.

Period of both $|\cos x|$ and $|\sin x|$ is π

$\therefore$ Period of $f(x)$ = L.C.M. of π and $\pi = \pi$ is false.

It's actual period is $\frac{\pi}{2}$ because

$$f(x+\pi/2) = |\cos(x+\pi/2)| + |\sin(x+\pi/2)|$$

$$= |-\sin x| + |\cos x| = |\sin x| + |\cos x|$$

$$\therefore f(x+\pi/2) = |\sin x| + |\cos x| = f(x)$$

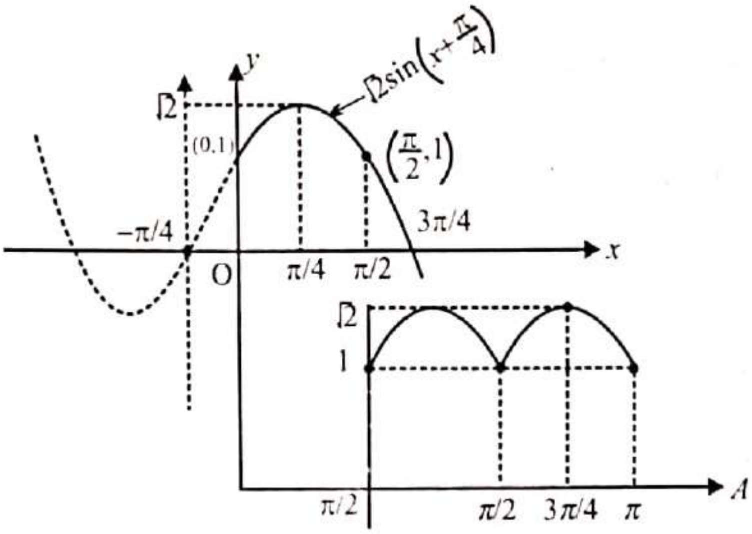
Illustration: $f(x) = |\sin x| + |\cos x|$

For $0 \le x < \frac{\pi}{2}$ $f(x) = \sin x + \cos x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ ...(A)

For $\frac{\pi}{2} \le x < \pi$ $f(x) = \sin x - \cos x$

$$= \sqrt{2}\sin\left(x-\frac{\pi}{4}\right) = \sqrt{2}\sin\left(\left(x-\frac{\pi}{2}\right)+\frac{\pi}{4}\right) = f\left(x-\frac{\pi}{2}\right)$$

The graph of this can be obtained by shifting the graph (A) rightward by $\frac{\pi}{2}$.



For $\pi \le x < \dfrac{3\pi}{2}$ $f(x) = \sin x - \cos x$

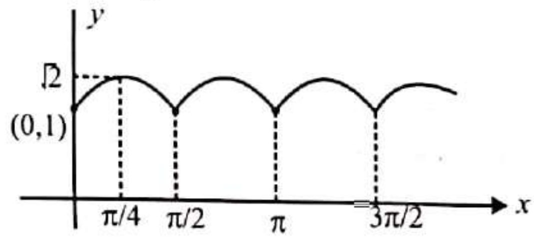
$$= -\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2}\sin\left(\left(x + \frac{\pi}{4}\right) - \pi\right)$$

$$= \sqrt{2}\sin\left((x-\pi) + \frac{\pi}{4}\right) = f(x-\pi)$$

(Shift (A) rightward by π to obtain its graph)

$f(x) = f\left(x - \dfrac{\pi}{2}\right) = f(x-\pi) = \ldots\ldots$ and so on

$\Rightarrow$ $f(x)$ is periodic with period $\dfrac{\pi}{2}$.



Now lets understand the cases when the result (v) fails. It is observed that if $g(x)$ and $f(x)$ are even functions, have same exponent and are co-functions of each other then the result (v) is definitely not applicable and it may also fail if the power of $g(x)$ and $f(x)$ are different. Such as:

$$f(x) = \sin^2 x + \cos^4 x$$
$$= \sin^2 x + \cos^2 x(1 - \sin^2 x)$$
$$= \sin^2 x + \cos^2 x - \sin^2 x\cos^2 x$$
$$= 1 - \frac{1}{4}\sin^2 2x = 1 - \frac{1}{4}\frac{(1-\cos 4x)}{2}$$
$$= 1 - \frac{1}{8} + \frac{1}{8}\cos 4x = \frac{7}{8} + \frac{1}{8}\cos 4x$$

Clearly the period $= \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

Hence, for the cases like this, it is advised to simplify $f(x)$ and then find the period.

Another failure case, we have learnt so far that if $f(x)$ and $g(x)$ are periodic functions with periods T_1 and T_2 respectively then $f(x)+g(x)$ is periodic with period equal to LCM of T_1 and T_2.

Now let us see another failure case of this rule, but first of all we should learn a very important property in number system. If some one asks to find the LCM of 1 and $\sqrt{2}$ or LCM of $\sqrt{2}$ and $\sqrt{3}$, what will be your answer? Students very often reply $\sqrt{2}$ and $\sqrt{6}$ respectively, which are false. Because LCM of numbers say x and y is divisible by x as well as y. But $\sqrt{2}$ here is not divisible by 1 in the 1st case. Hence, how $\sqrt{2}$ can be the LCM of 1 and $\sqrt{2}$? Certainly not. Exactly in the same way $\sqrt{6}$ is not the LCM of $\sqrt{2}$ and $\sqrt{3}$.

Remember that the LCM of a rational and an irrational number does not exist. Also, LCM of two different kinds of irrationals does not exist. But LCM of two similar irrational exists e.g. LCM of $2\sqrt{3}$ and $3\sqrt{3}$ is $6\sqrt{3}$. Hence, $f(x) + g(x)$ will be periodic if LCM of T_1 and T_2 exists.

e.g. $h(x) = (x - [x]) + \sin x$ is non periodic because periods of $(x - [x])$ and $\sin x$ are 1 and 2π respectively. But LCM of 1 and 2π does not exist.

Examples

Ex.1. Let $f(x)$ be a function satisfying $f(x+p) = 1 + \sqrt{2f(x) - (f(x))^2}$ $\forall x \in R$ $(P > 0)$. Examine whether $f(x)$ is periodic or not. If yes find its period.

Sol. Above relation is defined only

When $2f(x) - (f(x))^2 \geq 0 \Rightarrow 0 \leq f(x) \leq 2$

Also $f(x+p) = 1 + \sqrt{2f(x) - (f(x))^2} \geq 1 \Rightarrow f(x) \geq 1$

Hence, $1 \leq f(x) \leq 2$

Again $(f(x+p) - 1)^2 = 2f(x) - (f(x))^2$

$\Rightarrow (f(x+p) - 1)^2 = 1 - (f(x) - 1)^2$ ...(i)

Replacing x by $x + p$ we get

$(f(x+2p) - 1)^2 = 1 - (f(x+p) - 1)^2$ ...(ii)

Subtracting (i) from (ii) we get

$\Rightarrow \quad [f(x+2p) - 1]^2 = [f(x) - 1]^2$

$\Rightarrow \quad |f(x+2p) - 1| = |f(x) - 1|$

$\Rightarrow \quad f(x+2p) - 1 = f(x) - 1 \; (\because f(x) \geq 1)$

$\therefore \quad f(x+2p) = f(x) \Rightarrow f$ is periodic with period $2p$.

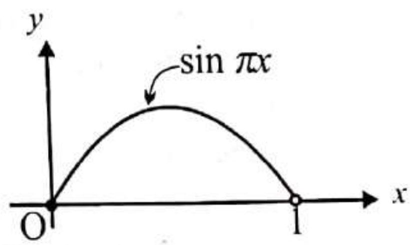
Ex.2. Find the period of $f(x) = \sin\left(\pi(x - [x])\right)$

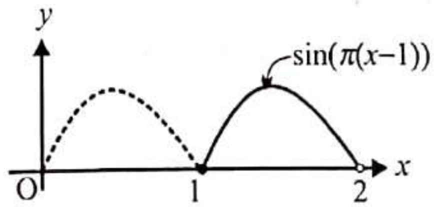
Sol. For $0 \leq x < 1$; $f(x) = \sin \pi x$

For $1 \leq x < 2$; $f(x) = \sin \pi(x-1) = f(x-1)$

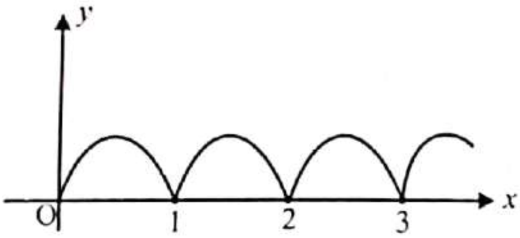
For $2 \leq x < 3$; $f(x) = \sin(\pi(x-2)) = f(x-2)$

and so on





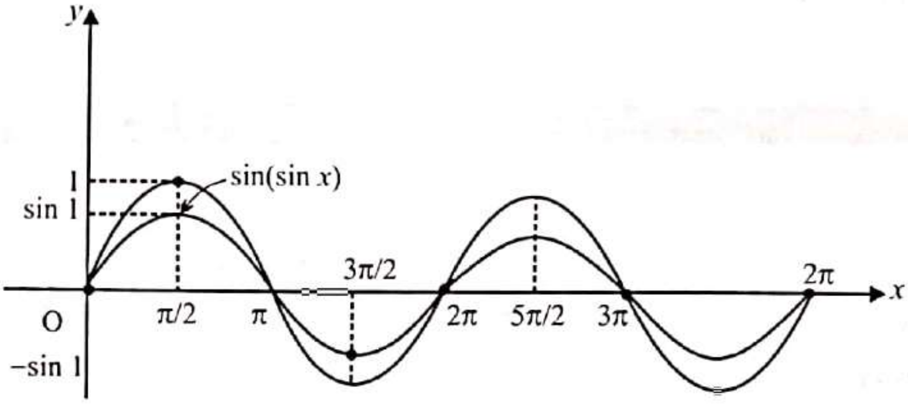
$\therefore \; f(x) = f(x-1) = f(x-2) = ...$



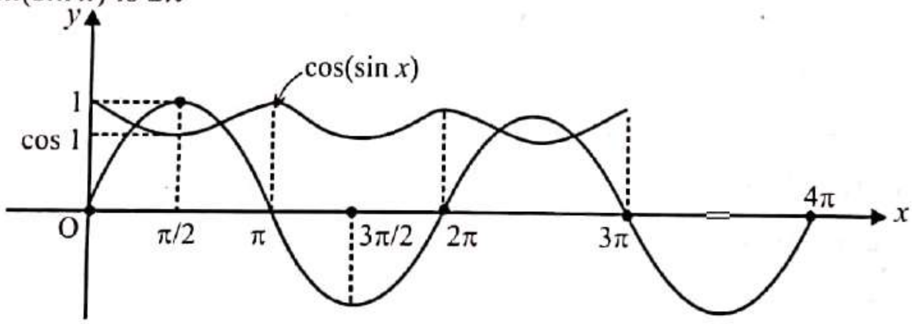
$\Rightarrow \; f(x)$ is periodic with period = 1

Ex.3. Find the period of $f(x) = \sin(\sin x)$

Sol.



Period of $\sin(\sin x)$ is 2π



Period of $\cos(\sin x)$ is π.

Ex.4. Let $f(x)$ be symmetric about the lines $x = 1$ and $x = 3$ then prove that $f(x)$ is a periodic function and find its period.

Sol. Since $f(x)$ is symmetric about the line $x = 1$;

$f(1 + t) = f(1 - t)$

Let $1 - t = x \;\; \Rightarrow \;\; t = 1 - x$

$\therefore \;\; f(2 - x) = f(x)$ ...(1)

Similarly, when $f(x)$ is symmetric about the line $x = 3$, we have

$f(6 - x) = f(x)$ ...(2)

Hence, $f(2 - x) = f(6 - x) = f((2 - x) + 4)$

$\Rightarrow \;\; f(X) = f(X + 4)$

Ex.5. If a function $f(x)$ satisfies the relation $f(x+1)+f(x-1)=\sqrt{3}f(x)$ $\forall x \in R$ prove that $f(x)$ is periodic and find its period.

Sol. Given that $f(x+1)+f(x-1)=\sqrt{3}f(x)$ ...(1)

Replacing x by $x-1$ and $x+1$ in (1) we get $f(x)+f(x-2)=\sqrt{3}f(x-1)$ ...(2)

and $f(x+2)+f(x)=\sqrt{3}f(x+1)$ ...(3)

Adding (2) and (3) we find

$$2f(x)+f(x-2)+f(x+2)=\sqrt{3}\left(f(x-1)+f(x+1)\right)=\sqrt{3}\sqrt{3}f(x)$$

$\Rightarrow$ $f(x-2)+f(x+2)=f(x)$ ...(4)

Now replacing x by $x+2$ we get $f(x)+f(x+4)=f(x+2)$ ...(5)

From (4) + (5) $f(x-2)+f(x+4)=0$ ...(6)

Replacing x by $x+6$ we get $f(x+4)+f(x+10)=0$ ...(7)

From (6) and (7) we find $f(x-2)=f(x+10)=f(x-2+12)$

Hence, $f(x)$ is periodic with period 12.