

Proving Functional Inequality:-

(i) $\sin n < n$ if $n > 0$

$$\text{Let } f(n) = \sin n - n$$

$$\therefore f'(n) = \cos n - 1 \leq 0 \text{ if } n \in \mathbb{R}$$

So, $f(n)$ is decr. of n if $n \in \mathbb{R}$

$$\text{thus } n > 0 \Rightarrow f(n) < f(0)$$

$$\text{if } n < 0 \Rightarrow f(n) > f(0)$$

$$\text{i.e. } \sin n < n, n > 0.$$

$$(ii) \sin n > 0, n < 0$$

$$(iii) \sin n > n, n > 0$$

$$(iv) \sin n < n, n < 0$$

$$(v) \tan n > n, \pi_2 > n > 0$$

$$(vi) \tan n < n, -\pi_2 < n < 0$$

$$(vii) \tan n < n, n > 0$$

$$(viii) \tan n > n, n < 0.$$

similarly, students must try to prove other inequalities (ii) to (viii).

Q. Prove that $f(n) = \frac{\ln(1+n)}{\ln(2+n)}$ is incr. for $n \in (0, \infty)$.

$$\text{Sol: } f'(n) = \frac{(2+n)\ln(2+n) - (1+n)\ln(1+n)}{(1+n)(2+n)\ln^2(2+n)}$$

$$\text{for } n > 0 \quad \ln(2+n) > \ln(1+n) > 0 \quad \& \quad (2+n) > (1+n) > 0$$

$$\therefore (2+n)\ln(2+n) > (1+n)\ln(1+n)$$

$$\text{So, } \boxed{f'(n) > 0}$$

i.e. incr. f :

Q. Show that $2\sin n + \tan n \geq 2n$ where $0 \leq n < \pi_2$

$$\text{Sol: } f(n) = 2\sin n + \tan n - 2n$$

$$\text{then } f'(n) = \frac{(\cos n - 1)^2(2\cos n + 2)}{\cos^2 n} \geq 0, n \in [0, \pi_2]$$

$$\therefore \text{for } n > 0 \Rightarrow f(n) \geq f(0)$$

$$\text{So, } 2\sin n + \tan n - 2n \geq 0$$

$$\text{i.e. } \boxed{2\sin n + \tan n \geq 2n}. \quad \text{Ans}$$

11.Q. prove that : $f(\theta) = \frac{\tan \theta}{\theta}$ is inv. of γ when $0 < \theta < \pi_2$. Also, prove that $\frac{\tan \eta_2}{\eta_2} > \frac{\tan \eta_1}{\eta_1}$ if $0 < \eta_1 < \eta_2 < \pi_2$.

$$\text{Soln: } f'(\theta) = \frac{2\theta - \sin 2\theta}{2(\theta^2 \cos^2 \theta)}$$

$$\therefore \sin 2\theta < 2\theta, \theta > 0.$$

$$\therefore f'(\theta) > 0 \quad (\text{inv. of } \gamma)$$

Now, if $\eta_1 < \eta_2 \Rightarrow f(\eta_1) < f(\eta_2)$

$$\text{So, } \boxed{\frac{\tan \eta_2}{\eta_2} > \frac{\tan \eta_1}{\eta_1}}$$

Proved

12.Q. let $a+b=4$, where $a < 2$ & let $g(m)$ be a differentiable fn. if $dg/dx > 0 \forall m$. prove that $\int_a^b g(m) dm + \int_a^b g(m) dx$ in $\text{an}(b-a)$ increases.

$$\text{Soln: let } b-a=t \quad \& \quad b+a=4 \quad (a > 2 \& b > 2)$$

$$\therefore b = \frac{t+4}{2} \quad \& \quad a = 4 - \frac{t}{2}$$

$$\text{Also, let } \phi(t) = \int_a^b g(m) dm + \int_a^b g(m) dx$$

$$\therefore \phi'(t) = \frac{1}{2} \left\{ g\left(\frac{t+4}{2}\right) - g\left(\frac{4-t}{2}\right) \right\}$$

so, $g(m)$ is inv. fn.

$$\text{for } g\left(\frac{t+4}{2}\right) > g\left(\frac{4-t}{2}\right) \Rightarrow \phi'(t) > 0$$

$\therefore \boxed{\phi(t) \text{ inv. an 't' inv.}}$

13.Q. find the int. in which the fn. $f(m) = \sin(\ln m) - \cos(\ln m)$.

$$\text{Soln: } f'(m) = \sqrt{m} \sin(\pi_4 + \ln m)$$

$$\therefore \sin(\pi_4 + \ln m) \geq 0 \Rightarrow 2n\pi \leq \pi_4 + \ln m \leq (2n+1)\pi$$

$$\text{So, } \boxed{m \in [e^{2n\pi - \pi_4}, e^{2n\pi + 3\pi_4}]}.$$

14.Q. find the set of all values of 'a' for which $f(m) =$

$$\left(\frac{\sqrt{9+4}}{(1-a)} - 1 \right) m^a - 3x + 4m \text{ dec. & neg}$$

$$\text{so, } f'(n) = 5 \left(\frac{\sqrt{n+4}}{n-9} - 1 \right) n^4 - 3$$

$\therefore f'(n) \leq 0 \forall n \in \mathbb{N}$.

$$\text{i.e. } \left(\frac{\sqrt{n+4}}{n-9} - 1 \right) \leq \left(\frac{3}{\sqrt{n+4}} \right)_{\min}, n \in \mathbb{N} \Rightarrow \left(\frac{\sqrt{n+4}}{n-9} - 1 \right) \leq 0$$

$$\text{i.e. } \frac{\sqrt{n+4}}{n-9} \leq 1 \Rightarrow \boxed{n \in \left[-4, \frac{3-\sqrt{51}}{2} \right] \cup (1, \infty)}.$$

15.Q Does the parameter 'a' possess any values for which the $f: f(n) = \left(1 - \frac{\sqrt{21-4n-n^2}}{n+1} \right) n^3 + 5n + \sqrt{6}$ is incr. $\forall n \in \mathbb{N}$
 → students must try this.

16.Q Let $f(n) = 2/\sqrt{3} \tan^{-1} \left(\frac{2n+1}{\sqrt{2}} \right) \log(n^2+n+1) + (b^2-5b+2)n + 9$

If $f(n)$ is decr. $f: f(n) \forall n \in \mathbb{N}$, find all possible values of 'b'.

$$\text{soln: } f'(n) = \frac{1}{n^2+n+1} - \frac{2n+1}{n^2+n+1} + b^2-5b+2 \leq 0 \forall n \in \mathbb{N}$$

$$\Rightarrow b^2-5b+2 \leq \frac{2n}{n^2+n+1} \forall n \in \mathbb{N}.$$

$$\therefore b^2-5b+2 \leq \left(\frac{2n}{n^2+n+1} \right)_{\min}.$$

$$\text{Let } 4 = \frac{2n}{n^2+n+1} \Rightarrow -2 \leq 4 \leq 2/\sqrt{3}$$

$$\therefore 4_{\min} = -2$$

$$\text{i.e. } b^2-5b+2 \leq -2 \Rightarrow \boxed{\frac{5-\sqrt{5}}{2} \leq b \leq \frac{5+\sqrt{5}}{2}}$$

17.Q Prove: $(a+b)^n \leq a^n + b^n$, $a > 0, b > 0 \notin 0 \leq a \leq 1$.

$$\text{Soln: } (a/b+1)^n \leq (a/b)^n + 1 \Rightarrow (n+1)^n \leq 1 + n^n \quad (n = a/b)$$

$$\text{Let } f(n) = (a/b)^n - n^n - 1 \Rightarrow f'(n) = n \left[\frac{1}{(1+a/b)^{n-1}} - \frac{1}{n^{n-1}} \right]$$

Consider, $1+a/b > n \Rightarrow (1+a/b)^{1-n} > n^{1-n} \quad (1-a/b > 0)$

$$\Rightarrow f'(n) < 0 \rightarrow \text{decr. } f(n)$$

$$\text{Consider } n > 0 \Rightarrow f(n) \leq f(0) \therefore \boxed{(a/b+1)^n \leq a^n + b^n}.$$

Range :-

Method to find the range of $y = f(x)$:-

first of all find the domain of $y = f(x)$,

(i) If domain is a set having only finite no. of points, range is the set of corr. $f(x)$ values.

(ii) If domain of $f(x)$ is \mathbb{R} or, $\mathbb{R} - \{ \text{some finite pts} \}$, or an infinite int., express 'x' in terms of 'y'. From this find 'y' for 'x' to be defd. or, real. or, form an eqn. in terms of 'x' and apply the condn. for real roots.

(iii) If domain is not an infinite interval, find the least and the greatest values of $f(x)$ using monotonicity.

Q. Find the range of $y = \frac{x^2}{1+x^2}$

Soln:- y is defd. & neR hence, domain is ' \mathbb{R} '

\therefore from $y = \frac{x^2}{1+x^2}$, we have

$$xy + y = x^2 \Rightarrow x^2 = \frac{y}{1-y}$$

$$\therefore y = \pm \sqrt{\frac{y}{1-y}}$$

$$\therefore \frac{y}{1-y} \geq 0 \Rightarrow \boxed{y \in [0, 1)}.$$

Q.19. Find the domain and range of the f: $y = \log_e(2x^2 - 4x + 5)$.

Soln:- $(2x^2 - 4x + 5) > 0 \forall x \in \mathbb{R}$.

$\therefore \Omega_f \in \mathbb{R}$.

Now, from $y = \log_e(2x^2 - 4x + 5)$ we have $(2x^2 - 4x + 5) = e^y$

$$\Rightarrow 2x^2 - 4x + (5 - e^y) = 0$$

$$\therefore \Delta \geq 0 \Rightarrow 4^2 - 4 \cdot 2(5 - e^y) \geq 0 \Rightarrow e^y \geq \frac{1}{2}$$

$$\therefore \boxed{y \in [\ln \frac{1}{2}, \infty)}$$

Q.20. Find the range of $y = a \cos \alpha + b \sin \alpha$.

Soln:- Now, $y = a \cos \alpha + b \sin \alpha$

$$= \sqrt{a^2+b^2} [\cos \alpha \cos \theta + \sin \alpha \sin \theta] \text{ where } \cos \theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$= \sqrt{a^2+b^2} \cos(\alpha - \theta)$$

$$\therefore \boxed{\text{Range} = [-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]}$$

Q.21. Find the range of $y = \frac{y-1}{y^2+2y}$.

Soln:- $\Omega_f \in R - \{-2, 0\}$

$$\text{Now, } y = \frac{y-1}{y+2}, y \neq 0$$

$$\Rightarrow x = \frac{1+y}{1-y} \quad \therefore y \text{ is real if } y-1 \neq 0 \Rightarrow y \neq 1.$$

$$\text{Also, because, } y = \frac{x-1}{x+2} \text{ for } y \neq 0$$

Hence, we have to exclude the value of y obt'l. when $y \rightarrow 0$

$$\& \lim_{y \rightarrow 0} \frac{y-1}{y+2} = -\frac{1}{2}$$

$$\therefore \text{Range is } \boxed{y \in R - \{-\frac{1}{2}, 1\}}$$

Why we have excluded $y = -\frac{1}{2}$, can be understood as follows.

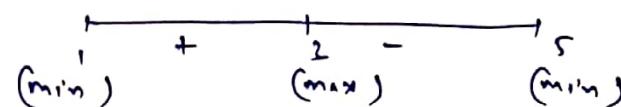
$$\text{As, } y \neq 0 \& y = \frac{1+y}{1-y} \text{ Hence, } \frac{1+y}{1-y} \neq 0$$

$$\Rightarrow 1+y \neq 0 \Rightarrow y \neq -1$$

Q.22. Find the range of : $y = \sqrt{n-1} + \sqrt{s-n}$

Soln:- $\Omega_f : n \in [1, 5]$

$$\therefore \frac{dy}{dn} = \frac{1}{2\sqrt{n-1}} - \frac{1}{2\sqrt{s-n}} \Rightarrow \frac{dy}{dn} = 0 \Rightarrow n = 2$$



$$\therefore \boxed{y \in [2, 2\sqrt{2}]}$$