

MONOTONICITY :-

Monotonic behaviour means 'sense of ascending or descending'.

(I) Increasing fⁿ :-

(a) Strictly increasing fⁿ :-

$$\text{If } x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Nature of derivative :-

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$x < x+h$, hence $f(x) < f(x+h)$

$$\therefore f(x+h) - f(x) > 0 \Rightarrow f'(x) = \text{+ve i.e. } f'(x) > 0$$

Note! - $f'(x)$ may be zero but only at finite no. of points & not in an interval.

(B) Non-decreasing fⁿ :- If $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

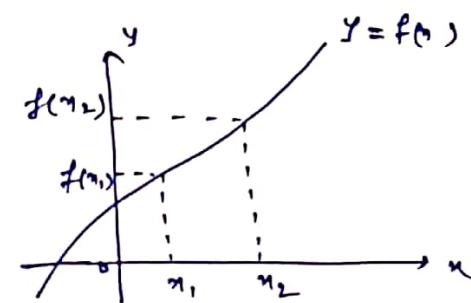
for portion AB & CD if $x_1 < x_2$

$$\Rightarrow f(x_1) < f(x_2)$$

$$\& \text{ for BC, } x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$$

$$\therefore f(x_1) \leq f(x_2)$$

$$\text{Hence, } f'(x) \geq 0.$$



(II) Decreasing fⁿ :-

(a) Strictly decreasing fⁿ :-

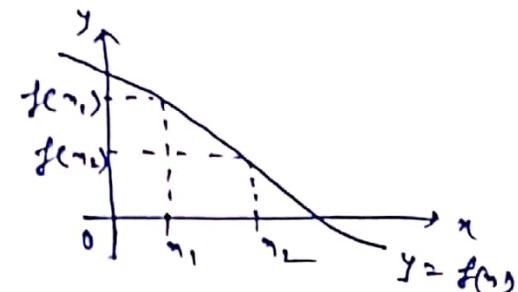
$$\text{follows, } x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

Nature of derivative,

$$\therefore f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore x < x+h \Rightarrow f(x) > f(x+h)$$

$$\text{i.e. } f'(x) = \text{-ve} \Rightarrow f'(x) < 0$$



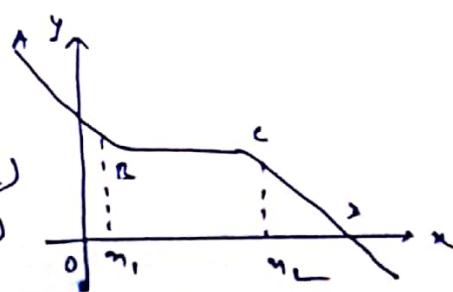
(b) Non-decreasing function :-

$$\text{if } x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$$

i.e. for portion AB & CD, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

while for portion BC, $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$

$$\therefore \text{Hence, } f'(x) \leq 0.$$



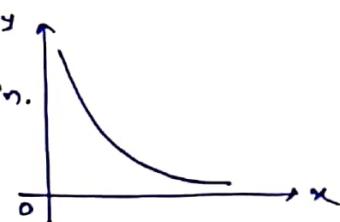
Properties of Monotonic function:-

- If $f(x)$ is inc. f' > 0 then $f'(x)$ is decr. f' < 0
 - If $f(x) \in g(x)$ are incr. fn then $f(x) + g(x)$ is also incr. f' > 0
 - If $f(x)$ is incr. f' in $[a, b]$ then the greatest & least value of $f(x)$ in $[a, b]$ are $f(b)$ & $f(a)$ resp.
 - If $f(x)$ & $g(x)$ both are monotonically incr. or decr. f' > 0 on $[a, b]$ then $gof(x)$ is monotonically incr. f' > 0 on $[a, b]$.
 - If one of the two fn $f(x)$ & $g(x)$ is st. incr. & other is st. decr. then gof is st. decr. on $[a, b]$.
 - If $f(x)$ is st. incr. on $[a, b]$ such that it is continuous, then f^{-1} is cont on $(f(a), f(b))$.
 - If $f(x)$ is st. incr. f' > 0 on $[a, b]$, then f^{-1} exists and it is also a st. incr. f' > 0
- proof:- Let, $f^{-1}(y) = h(y) \Rightarrow y = foh(y)$
- $$1 = f'(h(y)) \cdot h'(y)$$
- $$\therefore f'(y) > 0 \text{ (given)} \Rightarrow h'(y) > 0 \Rightarrow \text{incr.}$$

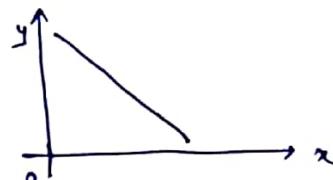
Classification of st. incr. fn on the basis of shape:-

(i) concave up or convex down:-

when, $f'(x) < 0$ & $f''(x) < 0 \forall x \in \text{domain.}$

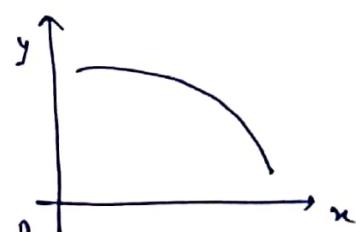


(ii) when $f'(x) < 0$ & $f''(x) = 0 \forall x \in \text{domain.}$



(iii) convex up, or concave down:-

when, $f'(x) < 0$ & $f''(x) > 0 \forall x \in \text{domain.}$



1.Q. Find the interval of inc. or decr. of the following $f(x)$:-

$$(a) f(x) = 2x^2 + 3x^2 - 12x + 1$$

$$\text{SOL: } f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x+2)(x-1).$$

$$\therefore f'(x) > 0 \text{ (for inc.)} \Rightarrow x \in (-\infty, -2) \cup (1, \infty)$$

$$\& f'(x) < 0 \text{ (for decr.)} \Rightarrow x \in (-2, 1).$$

$$(b) f(x) = \int_1^x (t^2 + 2t)(t^2 - 1) dt$$

$$\text{SOL: } f'(x) = (x^2 + 2x)(x^2 - 1) =$$

$$\text{for } f'(x) > 0 \Rightarrow x(x+2)(x+1)(x-1) > 0 \Rightarrow x \in (-\infty, -2) \cup (-1, 0) \cup (1, \infty)$$

$$\& f'(x) < 0 \Rightarrow x(x+2)(x+1)(x-1) < 0 \Rightarrow x \in (-2, -1) \cup (0, 1).$$

$$(c) f(x) = 2\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3, 0 \leq x \leq \pi$$

→ student must try this.

2.Q. Find the values of const a so that $f(x) = \sin x - ax + b$ decr. along the entire no. scale.

$$\text{SOL: } f'(x) = \cos x - a \leq 0 \Rightarrow a \geq (\cos x)_{\max} \Rightarrow a \geq 1$$

$$\& b \in \mathbb{R}.$$

3.Q. At what values of 'a' does the f : $f(x) = 2 + ax - x^2$ decr. $\forall x \in \mathbb{R}$.

$$\text{SOL: } f'(x) = a - 2x^2 \leq 0 \Rightarrow a \leq (2x^2)_{\min} \Rightarrow a \leq 0 \Rightarrow a \in (-\infty, 0].$$

$$4.Q. \text{ Let, } f(x) = \begin{cases} x e^{ax}; & x \leq 0 \\ x + ax^2 - x^3; & x > 0 \end{cases} \text{ where 'a' is a positive const}$$

Find the interval in which $f'(x)$ is inc.

$$\text{SOL: } f'(x) = \begin{cases} ae^{ax} + e^{ax}; & x \leq 0 \\ 1 + 2ax - 3x^2; & x > 0 \end{cases} \Rightarrow f''(x) = \begin{cases} (2a + a^2 x)e^{ax}; & x \leq 0 \\ 2a - 6x; & x > 0 \end{cases}$$

$$\therefore f'(x) = 0 \Rightarrow x = -2/a \text{ (in } x \leq 0) \& x = a/2 \text{ (in } x > 0)$$

$$\begin{array}{ccccccc} & - & + & & + & - & \\ \leftarrow & \hline & -2/a & & a/2 & & \rightarrow \end{array}$$

$$\therefore x \in (-2/a, a/2)$$

S.Q. Let $f: (0, \infty) \rightarrow \mathbb{R}$ be given by $f(n) = \int_{1/n}^n e^{-(t+1/t)} dt$ then

(a) $f(n)$ is incr. on $[1, \infty)$ (b) $f(n)$ is decr. on $(0, 1)$

(c) $f(n) + f(1/n) = 0 \quad \forall n \in (0, \infty)$ (d) $f(2^n)$ is an odd fn of n on \mathbb{R} .

$$\text{Sol: } f'(n) = \frac{1}{n} e^{-(n+1/n)}$$

$$\therefore f(n) + f(1/n) = \int_{1/n}^n e^{-(t+1/t)} \frac{dt}{t} + \int_n^{1/n} e^{-(t+1/t)} \frac{dt}{t} = 0$$

$$f(2^n) + f(2^{-n}) = 0 \Rightarrow f(2^n) = -f(2^{-n}) \Rightarrow \text{odd fn.}$$

So, options a, c

S.Q. If $f(n) = n \cos \frac{1}{n}, n > 1$ then,

(a) for all n , in the int $[1, \infty)$, $f(n+2) - f(n) < 2$.

(b) $\lim_{n \rightarrow \infty} f'(n) = 1$

(c) for all n , in the int $[1, \infty)$, $f(n+2) - f(n) > 2$.

(d) $f'(n)$ is cont. decr. in $[1, \infty)$

$$\text{Sol: } f'(n) = \cos \frac{1}{n} + \frac{1}{n} \sin \frac{1}{n} \Rightarrow \lim_{n \rightarrow \infty} f'(n) = 1.$$

$$\therefore f''(n) = -\frac{1}{n^2} \cos \left(\frac{1}{n}\right) < 0, \quad \forall n \in [1, \infty)$$

$$\text{Also, } f'(n) > \lim_{n \rightarrow \infty} f'(n)$$

$$\Rightarrow \frac{f(n+2) - f(n)}{(n+2) - n} > 1$$

So, options b, c, d

S.Q. Let, $f(n) = (1-n)^2 \sin^2 n + n^2 \quad \forall n \in \mathbb{R}$ then let $g(n) = \int_1^n \left(\frac{2(t-1)}{t+1} - \sin t \right) dt$ & $\forall n \in (1, \infty)$.

Consider the statements:-

P: There exist some $n \in \mathbb{R}$ such that $f(n) + 2n = 2(1+n^2)$

Q: There exists some $n \in \mathbb{R}$ such that $2f(n) + 1 = 2n(1+n)$ then

(a) P & Q are true (b) P is true & Q is false

(c) P is false & Q is true (d) P & Q are false

$$\text{S.Q. :- } f(n) + 2n = 2(1+n^2)$$

$$\text{i.e. } (1-n^2) \sin^2 x + n^2 + 2n = 2(1+n^2)$$

or $(1-n^2) \cos^2 x = -1$, which is not possible $\forall n \in \mathbb{R}$.

$\therefore P \rightarrow \text{false.}$

$$\text{Let, } H(n) = 2f(n) + 1 - 2n(1+n)$$

$$\therefore H(0) = 2f(0) + 1 - 0 = 1 \quad \& \quad H(1) = 2f(1) - 4 + 1 = -1$$

$\therefore H(n)$ has a solution in $(0, 1)$

i.e. $Q \rightarrow \text{True.}$ So, option 'c'.

7. Q. Which of the following is True?

(a) g is incr. on $(1, \infty)$ (b) g is decr. on $(1, \infty)$

(c) g is decr. on $(1, 2)$ & decr. on $(2, \infty)$

(d) g is decr. on $(1, 2)$ & incr. on $(2, \infty)$.

$$\text{S.Q. :- } g'(n) = \left[\frac{2(n-1)}{n+1} - \ln n \right] f(n) \Rightarrow f(n) > 0 \quad \& \quad n \in (1, \infty)$$

$$\text{Also, let } h(n) = \frac{2(n-1)}{n+1} - \ln n$$

$$\therefore h'(n) = -\frac{(n+1)^2}{(n+1)^2} < 0, \quad n \in (1, \infty)$$

$\therefore h(n)$ is decr. $\forall n$ for $n > 1$.

Hence $g'(n) < 0 \quad \& \quad n \in (1, \infty)$.

So, option 'b'

8. Q. Let $f: [0, 1] \rightarrow \mathbb{R}$ be a fcn. suppose the f : f' is twice diff.

$f(0) = f(1) = 0$ & satisfies $f''(n) - 2f'(n) + f(n) \geq e^n, \quad n \in [0, 1]$

which of the following is true for $0 < n < 1$?

(a) $0 < f(n) < \infty$ (b) $-1/2 < f(n) < 1/2$ (c) $-1/4 < f(n) < 1$

(d) $-\infty < f(n) < 0$.

$$\text{Hint:- } f''(n) - 2f'(n) + f(n) \geq e^n \Rightarrow [f''(n) - f'(n)] - [f'(n) - f(n)] \geq e^n$$

$$\Rightarrow \frac{d}{dn} (e^{-n} f'(n)) - \frac{d}{dn} (e^{-n} f(n)) \geq 0 \Rightarrow \frac{d}{dn} [\frac{d}{dn} (e^{-n} f(n))] \geq 1.$$

Let $g(n) = e^{-n} f(n) \Rightarrow g''(n) \geq 1 > 0 \rightarrow g(n)$ is concave up.

\therefore option 'd'