

### Functional Eqn :-

functional eqns are eqns where unknowns are functions, rather than a traditional variable. However the methods used to solve functional eqns can be quite different than the methods for isolating a traditional variable. For ex:-  $f(m) - f(0) = m \cdot y$  is a functional eqn.

### Questions based on functional eqn:-

(i.) If  $f(n)$  be a polynomial funcn satisfying  $f(n) \cdot f(1/n) = f(n) + f(1/n)$  &  $f(4) = 65$ . Find  $f(5)$ .

$$\text{Soln: } f(1/n) (f(n)-1) = f(n) \Rightarrow f(1/n) (f(n)-1) - f(n)+1 = 1$$

$$\therefore (f(1/n)-1) (f(n)-1) = 1 \quad \dots \quad (i)$$

eqn (i) can only hold if  $f(n) = \pm n^n + 1$

$$\text{Also, } f(4) = 64+1 = 4^3+1 \Rightarrow \boxed{f(n) = n^n+1}$$

$$\text{So, } \boxed{f(5) = 126}$$

(ii.) Let  $f_1(n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ , then  $f_1(1) + f_1(2) + f_1(3) + \dots + f_1(n)$  is equal to ---

- (a)  $n f_1(n) - 1$       (b)  $(n+1) f_1(n) + n$       (c)  $n f_1(n) + n$       (d)  $n f_1(n) + n$ .

Soln:- In  $f_1(n) = f_1(1) + f_1(2) + f_1(3) + \dots + f_1(n)$

In  $f_1(1) + f_1(2) + f_1(3) + \dots + f_1(n)$ ,

$1$  occurs of ' $n$ ' times,  $\frac{1}{2}$  occurs  $(n-1)$  times,  $\frac{1}{3}$  occurs  $(n-2)$  times

& so on ---

$$\therefore f_1(1) + f_1(2) + f_1(3) + \dots + f_1(n) = n \cdot 1 + (n-1) \frac{1}{2} + (n-2) \cdot \frac{1}{3} + \dots + 1 \cdot \frac{1}{n}$$

$$= n f_1(n) - \left[ (1 - \frac{1}{2}) + (1 - \frac{1}{3}) + (1 - \frac{1}{4}) + \dots + (1 - \frac{1}{n}) \right]$$

$$= n f_1(n) - [n - f_1(n)]$$

$$= (n+1) f_1(n) - n$$

$$\text{So, } \boxed{\text{option 'b'}}$$

(iii) Let  $f: R \rightarrow R$  be a continuous fn. such that  $f(n) - 2f(\gamma_2) + f(\gamma_4) = n^2$ , then  $f(2) = \dots$

- (a)  $f(0)$     (b)  $f(0) + 4$     (c)  $9 + f(0)$     (d)  $12 + f(0)$

Soln:- Here,  $f(n) - 2f(\gamma_2) + f(\gamma_4) = n^2$

$$\therefore f(\gamma_2) - 2f(\gamma_4) + f(\gamma_8) = (\gamma_2)^2$$

$$f(\gamma_4) - 2f(\gamma_8) + f(\gamma_{16}) = (\gamma_4)^2$$

.....

$$f\left(\frac{n}{2^n}\right) - 2f\left(\frac{n}{2^{n+1}}\right) + f\left(\frac{n}{2^{n+2}}\right) = (\gamma_2)^2$$

Adding all,

$$f(n) - f(\gamma_2) - f\left(\frac{n}{2^{n+1}}\right) + f\left(\frac{n}{2^{n+2}}\right) = n^2 \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{2n}}\right)$$

As  $n \rightarrow \infty$ , we get  $f(n) - f(\gamma_2) = 4n^2/2$

Similarly, repeating same process again, we get

$$f(n) - f(0) = 16n^2/4$$

$$\text{i.e. } f(2) = 16 + f(0)$$

$\therefore$  option 'd'.

(iv) If a fn. satisfies  $(n-y)f(n+y) - (n+y)f(n-y) = 2(n^2y - y^2)$ ,  $\forall n, y \in \mathbb{R}$ . &  $f'(1) = 2$  then,

(a)  $f(n)$  must be polynomial &  $f(2) = 12$     (b)  $f(0) = 0$

(c)  $f(n)$  may not be diff.

Soln:- Here,  $f(n+y) \times (n-y) - (n+y)f(n-y) = 2y(n-y)(n+y)$

$$\text{Let, } n-y = p \quad \& \quad n+y = q$$

$$\therefore pqf(q) - qr(p) = 2pq(n-p) \Rightarrow \frac{f(n)}{q} - \frac{f(n)}{p} = n-4$$

$$\text{or } \left(\frac{f(n)}{q} - n\right) = \left(\frac{f(n)}{p} - n\right) = \text{const.}$$

$$\text{Let, } f(n)/n - n = \lambda \Rightarrow f(n) = \lambda n + n^2$$

$$\Rightarrow f(1) = 2 \Rightarrow \lambda + 1 = 2 \Rightarrow \lambda = 1$$

$$\therefore f(n) = n^2 + n \quad \text{So, option a, b, c.}$$

(v) Consider a function  $f(n)$  defined for all  $n \in \mathbb{N}$ . If  $f(n)$  satisfies the following two conditions.

$$(a) f(1) + f(2) + f(3) + \dots = 1$$

$$(b) f(n) = \{(1-p)p^{-1}\} \{f(n+1) + f(n+2) + \dots\}, \text{ where } 0 < p < 1. \text{ Then } f(2) = \dots$$

- (A)  $p(1-p)$  (B)  $1-p$  (C)  $1+p$  (D) NOT.

$$\text{S.Q.} \therefore f(n) = \{(1-p)p^{-1}\} \{f(n+1) + f(n+2) + \dots\}$$

for  $n=1$ ,

$$f(1) = \{(1-p)p^{-1}\} \{f(2) + f(3) + \dots\}$$

$$= \{(1-p)p^{-1}\} \{1 - f(1)\}$$

$$= (1-p)p^{-1} - (1-p)p^{-1}f(1)$$

$$\Rightarrow f(1) \{1 + (1-p)p^{-1}\} = (1-p)p^{-1}$$

$$\text{or, } f(1)p^{-1} = (1-p)p^{-1}$$

for  $n=2$

$$f(2) = \{(1-p)p^{-1}\} \{f(3) + f(4) + \dots\} = \{(1-p)p^{-1}\} \{1 - f(2)\}$$

$$= (1-p) - (1-p)p^{-1}f(2) = \{1 + (1-p)p^{-1}\} = 1-p$$

$$\therefore f(2) = p(1-p)$$

$\therefore$ , option 'A'

(vi) If  $2f(m) = f(ny) + f(n'y)$  for all positive values of  $n \neq n'$ , then if  $f(0) = 0$  &  $f'(0) = 1$ , then  $f(e) = \dots$

S.Q. put,  $n=1$ , in  $2f(2) = f(ny) + f(n'y) \dots \text{(i)}$

$$\therefore 2f(2) = f(y) + f(1'y) \dots \text{(ii)}$$

Replacing ' $m'$  by ' $y$ ' in (i)

$$2f(2) = f(1y) + f(1'y) \dots \text{(iii)}$$

from (i) & (iii)

$$2\{f(m) - f(2)\} = f(n'y) - \{-f(n'y)\} = 2f(n'y)$$

$$\therefore f(m) - f(2) = f(n'y).$$

$$\therefore \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = f'(1) = 1$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(1+h)}{h} = 1 \text{ or } f(1) = 0$$

$$\therefore f'(n) = \lim_{h \rightarrow 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0} \frac{f(1+nh)}{h} = nh$$

$$\therefore f(n) = \log |n| + c$$

$$f(1) = 0 \Rightarrow c = 0 \Rightarrow f(n) = \log |n|$$

$$\text{i.e. } f(e) = 1.$$

(ii) Let  $f$  be a function from the set of possible int. to the set of real nos. such that,

$$(a) f(1) = 1$$

$$(b) \sum_{r=1}^n rf(r) = n(n+1)f(n), \forall n \geq 2, \text{ then find the value of } 212f(1062).$$

$$\text{Say } \therefore f(1) + 2f(2) + 3f(3) + \dots + nf(n) = n(n+1)f(n)$$

$$f(1) + 2f(2) + \dots + (n+1)f(n+1) = (n+1)(n+2)f(n+1)$$

$$\Rightarrow nf(n) = (n+1)f(n+1)$$

$$\text{i.e. } 2f(2) = 3f(3) = \dots = nf(n)$$

$$\Rightarrow f(n) = \frac{1}{2n}$$

$$\Rightarrow 212f(1062) = 1$$

$$(viii) If  $f^2(n) \cdot f\left(\frac{1-\gamma}{1+\gamma}\right) = \gamma^2 [n \neq -1; 1 \neq f(n) \neq 0]$ , then find  $|[f(-2)]|$ .$$

$$\text{Say } \therefore f^2(n) \cdot f\left(\frac{1-\gamma}{1+\gamma}\right) = \gamma^3 \quad (i)$$

$$\gamma \rightarrow \frac{1-\gamma}{1+\gamma}$$

$$\therefore f^2\left(\frac{1-\gamma}{1+\gamma}\right) f(n) = \left(\frac{1-\gamma}{1+\gamma}\right)^2 \quad (ii)$$

$$\text{by using (i) \& (ii) } \Rightarrow f^2(n) = \left(\frac{1+\gamma}{1-\gamma}\right)^2 n^2 \Rightarrow f(n) = \gamma^2 \left(\frac{1+\gamma}{1-\gamma}\right)$$

$$\therefore f(-2) = \frac{1}{2}$$

$$\Rightarrow |[f(-2)]| = 2$$

Students must try :-

- (i.) If  $f(n) = f(f(n)) (1+f(n)) = -f(n)$ , find  $f(3)$ .
- (ii.) If  $f(m+n, m+n) = mn$ , find any of  $f(m, n) \neq f(n, m)$ .
- (iii.) In a  $f$ :  $f(n)$ ,  $f(1) = 1$  &  $f(2n) = n + f(2n-1)$ , find  $f(200)$ .
- (iv.) If  $f$  is defined as  $f(n) = \left[ \log_{1/n} \left[ \frac{1}{1/n} \right] \right]$  where  $|n| \neq 1$  &  $f(1) = 0$ .  
when  $|n| = 1/m$ ,  $n \in \mathbb{N}$  then find  $f(n)$  where  $|n| \neq 1$ .
- (v.)  $f(n) = [1 + \sin n] + [2 + \sin 2n] + [3 + \sin 3n] + \dots + [n + \sin nn]$ ,  
 $n \in (0, \pi)$ .
- (vi.) If  $f(n) = \begin{cases} (r_2)^n, & n \geq 2 \\ f(n+1), & n < 4 \end{cases}$  then  $f(2 + \log_2 \frac{3}{2}) = ?$
- (vii.) If  $[3_n] + [4_n] = 5$  then  $n = \dots$  (C.I. → G.I.F.)
- (viii.) Let  $f(n) = n^2 + 2n - 2$ ,  $n \geq 0$ . If 'n' points  $x_1, x_2, x_3, \dots, x_n$  are  
so chosen on the x-axis such that  
(a)  $\sum_{i=1}^n f^{-1}(x_i) = f\left(\sum_{i=1}^n x_i\right)$   
(b)  $\sum_{i=1}^n f^{-1}(x_i) = \sum_{i=1}^n x_i$ , then any of 'n' is ...
- (ix.) Let  $f(n) = 1 - n - n^2$ . Then the real values of 'n' satisfying the  
inequality  $1 - f(n) - f^2(n) > f(1 - 5n)$  are ...
- (x.) Total no. of sol'n. of  $2^{\cos n} = |\sin n|$  in  $[-2\pi, 2\pi]$  is equal to ...
- (xi.) If  $f(n) = \cos[\pi^2]n + \cos[-\pi^2]n$ ; where C.I stand for G.I.F.  
Evaluate :  $f(\pi/4) = \dots$
- (xii.) If  $f(n) = \sin^m n + \sin^m (n + \pi/3) + \cos n \cdot \cos(n + \pi/3)$  then find  $f(\pi/4)$ .
- (xiii.) Find the domain of :  $f(x) = \log \left[ \log_{1/\sin x} (x^2 - 8x + 23) - \frac{3}{\log_2 |\sin x|} \right]$
- (xiv.) Find the domain of :  $f(n) = \sin^{-1} \left( \frac{[n]}{f(n)} \right)$
- (xv.) Find the domain of :  $f(n) = \sqrt{[n] - 1 + n^2}$